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NEW SCHEMES FOR THE FINITE-ELEMENT DYNAMIC ANALYSIS OF PIEZOELECTRIC DEVICES†

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New finite-element schemes are proposed for investigating harmonic and non-stationary problems for composite elastic and piezoelectric media. These schemes develop the techniques for the finite-element analysis of piezoelectric structures based on symmetric and partitioned matrix algorithms. In order to take account of attenuation in piezoelectric media, a new model is used which extends the Kelvin model for viscoelastic media. It is shown that this model enables the system of finite-element equations to be split into separate scalar equations. The Newmark scheme in a convenient formulation, which does not explicitly use the velocities and accelerations of the nodal degrees of freedom, is employed for the direct integration with respect to time of the finite-element equations of non-stationary problems. The results of numerical experiments are presented which illustrate the effectiveness of the proposed techniques and their implementation in the ACELAN finite-element software package.

1. MODELS FOR TAKING ACCOUNT OF ATTENUATION IN PIEZOELECTRIC STRUCTURES

In modern finite-element packages the Rayleigh method is used, as a rule, to take account of the attenuation in composite, solid structures, in which the damping matrix $\mathbf{C}_{uu}$ is formulated in the form

$$\mathbf{C}_{uu} = \sum_j (\alpha_{dj}\mathbf{M}_{uuj} + \beta_{dj}\mathbf{K}_{uuj}) \tag{1.1}$$

where $\mathbf{M}_{uuj}$ and $\mathbf{K}_{uuj}$ are the mass and stiffness matrices for the medium Ω_j, α_{dj} and β_{dj} are non-negative attenuation factors. Relation (1.1) is obtained assuming the Kelvin model for taking account of the loss in the medium Ω_j

$$\boldsymbol{\sigma} = \mathbf{c}_j \cdot\cdot(\boldsymbol{\varepsilon} + \beta_{dj}\dot{\boldsymbol{\varepsilon}}) \tag{1.2}$$

and the addition of the term $\alpha_{dj}\rho_j\dot{\mathbf{u}}$ to the inertial term $\rho_j\ddot{\mathbf{u}}$ in the equations of motion. Here, $\boldsymbol{\sigma}$ and $\boldsymbol{\epsilon}$ are the second-rank stress and deformation vectors, $\mathbf{c}_j$ is the semi-symmetric, fourth-rank modulus of elasticity tensor, ρ_j is the density and $\mathbf{u}$ is the displacement vector.

In the subsequent dynamic finite-element analysis of solid structures it is often found to be convenient to use the method of expansion with respect to the oscillation modes $\mathbf{W}_k$, which has been orthonormalized with respect to the mass matrix $\mathbf{M}_{uu} = \Sigma_j\mathbf{M}_{uuj}$ and are orthogonal with respect to the stiffness matrix $\mathbf{K}_{uu} = \Sigma_j\mathbf{K}_{uuj}$,

$$\mathbf{W}_m^* \cdot \mathbf{M}_{uu} \cdot \mathbf{W}_k = \delta_{mk}, \quad \mathbf{W}_m^* \cdot \mathbf{K}_{uu} \cdot \mathbf{W}_k = \omega_m^2\delta_{mk} \tag{1.3}$$

where δ_{mk} is the Kronecker delta and ω_m is the natural frequency corresponding to the eigenvector $\mathbf{W}_m$.

If, in relation (1.1)

$$\alpha_{dj} = \alpha_d, \quad \beta_{dj} = \beta_d \text{ for any } j, \tag{1.4}$$

then, by virtue of equalities (1.3), the vectors $\mathbf{W}_k$ turn out to be orthogonal with respect to the damping matrix $\mathbf{C}_{uu}$

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$$\mathbf{W}_m^* \cdot \mathbf{C}_{uu} \cdot \mathbf{W}_k = 2\xi_{dm}\omega_m\delta_{mk}, \quad \xi_{dm} = \frac{1}{2\omega_m}\alpha_d + \frac{\omega_m}{2}\beta_d$$

The coefficients ξ_{dm} are referred to as the attenuation factors of the modes. They are associated in a simple manner with the mechanical Q-factor for the individual modes: $\mathbf{Q}_m = 1/(2\xi_{dm})$.

We will now extend the Rayleigh method for taking account of attenuation, which has been described above, to structures containing elastic and piezoelectric media. In the case of elastic media $\Omega_j = \Omega_{ej}$, we shall, as before, accept the defining relations (1.2). In the case of piezoelectric media $\Omega_j = \Omega_{pj}$, we shall assume that the mechanical stress tensor σ and the electric induction vector $\mathbf{D}$ are related to the deformation tensor $\boldsymbol{\epsilon}$ and the electric field strength vector $\mathbf{E}$ by the equations

$$\boldsymbol{\sigma} = \mathbf{c}_j^E \cdot\cdot (\boldsymbol{\epsilon} + \beta_{dj}\dot{\boldsymbol{\epsilon}}) - \mathbf{e}_j^* \cdot \mathbf{E} \tag{1.5}$$

$$\mathbf{D} + \zeta_d\dot{\mathbf{D}} = \mathbf{e}_j \cdot\cdot (\boldsymbol{\epsilon} + \zeta_d\dot{\boldsymbol{\epsilon}}) + \boldsymbol{\epsilon}_j^S \cdot \mathbf{E} \tag{1.6}$$

where $\mathbf{e}_j$ is the third-rank piezo-moduli tensor, ε_j^S is the second-rank permittivity tensor and $\zeta_d \geq 0$ is the attenuation factor which reflects the electrical losses.

This model extends the Kelvin model (1.2) to the case of piezoelectric media. When $\zeta_d = 0$ in Eq. (1.6), we have the more particular model for taking account of attenuation in piezoelectric media which is adopted in several well-known finite-element packages such as ANSYS [1] and COSMOS/M [2]. When $\zeta_d = 0$ in Eqs (1.5) and (1.6), it is as if only mechanical damping is taken into account. It is true that, by virtue of the coupled state of the mechanical and electric fields, the attenuation effects will also extend into the electric fields when $\zeta_d = 0$.

The main inadequacy of model (1.5), (1.6) when $\zeta_d = 0$ lies in the fact that, in the implementation of the method of expansion in modes, the system of finite-difference equations will not be split into independent equations for the individual modes, since the eigenvectors $\mathbf{W}_k$ will not be orthogonal with respect to the damping matrix $\mathbf{C}_{uu}$ which is obtained. As a result, to split the equations it is necessary to use a special form of the damping matrix which does not follow from model (1.5), (1.6) when $\zeta_d = 0$ [3, 4]. Model (1.5), (1.6) when $\beta_{dj} = \zeta_d$, which has been analysed for some time past [5, 6], removes this shortcoming. It has been shown [5] that the model obtained satisfies conditions which ensure the dissipation of energy, and the possibility of splitting the finite-element system into independent equations for the modes has been demonstrated [6] for the case of harmonic oscillations.

The essence of model (1.5), (1.6) when $\beta_{dj} = \zeta_d$ can be seen more clearly from the different form of the governing relations connecting the pair $\{\varepsilon, \mathbf{D}\}$ with the pair $\{\sigma, \mathbf{E}\}$,

$$\boldsymbol{\sigma} = \mathbf{c}_j^D \cdot\cdot (\boldsymbol{\epsilon} + \zeta_d\dot{\boldsymbol{\epsilon}}) - \mathbf{h}_j^* \cdot (\mathbf{D} + \zeta_d\dot{\mathbf{D}}) \tag{1.7}$$

$$\mathbf{E} = -\mathbf{h}_j \cdot\cdot (\boldsymbol{\epsilon} + \zeta_d\dot{\boldsymbol{\epsilon}}) + \boldsymbol{\beta}_j^S \cdot (\mathbf{D} + \zeta_d\dot{\mathbf{D}}) \tag{1.8}$$

where

$$\boldsymbol{\beta}_j^S = (\boldsymbol{\epsilon}_j^S)^{-1}, \quad \mathbf{h}_j = \boldsymbol{\beta}_j^S \cdot \mathbf{e}_j, \quad \mathbf{c}_j^D = \mathbf{c}_j^E + \mathbf{e}_j^* \cdot \boldsymbol{\beta}_j^S \cdot \mathbf{e}_j$$

Hence, both the model adopted in ANSYS as well as model (1.7), (1.8), which allow the system of finite-element equations to the split, follow as special cases from model (1.5), (1.6).

We note that, in the case of piezoelectric media, it is also possible to use integral models to take account of the attenuation, which generalize the usual integral models of viscoelastic media. However, such approaches in the finite-element analysis of non-stationary problems lead to extremely complex computational schemes [7]. The almost complete lack of experimental data on the relaxation moduli of piezoelectric media also prevents the use of integral models of attenuation.

2. FINITE-ELEMENT EQUATIONS FOR ANALYSING PIEZOELECTRIC DEVICES

We shall assume that a piezoelectric device is a solid Ω consisting of N homogeneous domains Ω_j ($j = 1, 2, \ldots, N$), generally speaking, with different piezoelectric or elastic properties. We shall assume that, in the domains $\Omega_j = \Omega_{pj}$ with piezoelectric properties, the displacement vector $\mathbf{u}(\mathbf{x}, t)$ and the electric potential $\varphi(\mathbf{x}, t)$ satisfy the system of equations ($\mathbf{f}_j$ is the mass force density vector)

$$\rho_j \ddot{\mathbf{u}} + \alpha_{dj}\rho_j \dot{\mathbf{u}} - \nabla\cdot\boldsymbol{\sigma} = \mathbf{f}_j \tag{2.1}$$

$$\nabla\cdot\mathbf{D} = 0 \tag{2.2}$$

for the governing relations (1.5), (1.6) and the formulae

$$\boldsymbol{\epsilon} = (\nabla\mathbf{u} + \nabla\mathbf{u}^*)/2, \quad \mathbf{E} = -\nabla\varphi \tag{2.3}$$

The usual equations (2.1) and (1.2), taking attenuation into account in accordance with Rayleigh's method, are taken for the domains $\Omega_j = \Omega_{ej}$, filled with elastic materials without piezoelectric properties.

The formulations of non-stationary problems of electro-elasticity have to be supplemented with the boundary conditions on the external boundary $\Gamma = \partial\Omega$; $\Omega = \cup_j\Omega_j$ and the conditions of rigid contact on the boundaries $\Gamma_{il} = \Gamma_i \cap \Gamma_l$ of the adjoining media Ω_i and Ω_l. Here, we shall permit all the basic types of boundary conditions of electro-elasticity including free electrodes and electrodes which are powered by current generators [8]. In the case of non-stationary problems it is also necessary to specify the initial conditions for the displacements and velocities

$$\mathbf{u}(\mathbf{x}, +0) = \mathbf{u}_0(\mathbf{x}), \quad \dot{\mathbf{u}}(\mathbf{x}, +0) = \dot{\mathbf{u}}_0(\mathbf{x}) \tag{2.4}$$

In order to solve initial-boundary-value problems of electro-elasticity, we shall use the finite element method in the classical Lagrangian formulation. We choose a matching mesh of finite elements in the domains Ω_{hj} which approximate the domains Ω_j. In this mesh of finite elements, we approximate the unknown field functions $\mathbf{u}$ and φ in the form

$$\mathbf{u}(\mathbf{x}, t) \approx \mathbf{N}_u^*(\mathbf{x})\cdot\mathbf{U}(t), \quad \varphi(\mathbf{x}, t) \approx \mathbf{N}_\varphi^*(\mathbf{x})\cdot\boldsymbol{\Phi}(t) \tag{2.5}$$

where $\mathbf{N}_u^*$ is the matrix of the form functions for the displacement field, $\mathbf{N}_\varphi^*$ is a row vector of the form functions for the potential field and $\mathbf{U}(t)$ and $\boldsymbol{\Phi}(t)$ are the global vectors of the nodal displacements and potentials.

The standard semi-discrete finite-element approximation of the generalized formulations of non-stationary problems (2.1)–(2.4), (1.5), (1.6), including the basic, main and natural boundary conditions, leads to the following system of differential equations

$$\mathbf{M}_{uu}\cdot\ddot{\mathbf{U}} + \mathbf{C}_{uu}\cdot\dot{\mathbf{U}} + \mathbf{K}_{uu}\cdot\mathbf{U} + \mathbf{K}_{u\varphi}\cdot\boldsymbol{\Phi} = \mathbf{F}_u \tag{2.6}$$

$$\zeta_d \mathbf{K}_{u\varphi}^*\cdot\dot{\mathbf{U}} + \mathbf{K}_{u\varphi}^*\cdot\mathbf{U} - \mathbf{K}_{\varphi\varphi}\cdot\boldsymbol{\Phi} = \mathbf{F}_\varphi + \zeta_d\dot{\mathbf{F}}_\varphi \tag{2.7}$$

with the initial conditions

$$\mathbf{U}(0) = \mathbf{U}_0, \quad \dot{\mathbf{U}}(0) = \dot{\mathbf{U}}_0 \tag{2.8}$$

which are obtained from the corresponding continual conditions (2.4).

The finite-element mass matrix $\mathbf{M}_{uu}$, damping matrix $\mathbf{C}_{uu}$ and stiffness matrix $\mathbf{K}_{uu}$ are the same as in the structural analysis. In particular, in the case of the damping matrix we have representation (1.1). All of these matrices are symmetric and non-negative definite and, moreover, $\mathbf{M}_{uu} > 0$. The finite-element matrices $\mathbf{K}_{u\varphi}$ and $\mathbf{K}_{\varphi\varphi}$ are due to the piezoelectric effect and reflect the piezoelectric and dielectric properties. Moreover, the matrix $\mathbf{K}_{\varphi\varphi}$ is symmetric and non-negative definite: $\mathbf{K}_{\varphi\varphi} \geqslant 0$. The vectors $\mathbf{F}_u$ and $\mathbf{F}_\varphi$ are formed as a result of taking account of the mechanical and electrical effects. Here, the main boundary conditions require additional transformations of the systems of finite-element equations (2.6), (2.7). We shall assume that such transformations in (2.6) and (2.7) are carried out using the techniques in [9], which preserve the structure of the finite-element matrices.

We note that, in the case of harmonic problems, when external actions vary as $\exp[i\omega t]$, it is obvious from (2.6) and (2.7) for the amplitude values, that we have a system of linear algebraic equations with the symmetric matrix

$$-\omega^2\mathbf{M}_{uu}\cdot\mathbf{U} + i\omega\mathbf{C}_{uu}\cdot\mathbf{U} + \mathbf{K}_{uu}\cdot\mathbf{U} + \mathbf{K}_{u\varphi}\cdot\boldsymbol{\Phi} = \mathbf{F}_u \tag{2.9}$$

$$\mathbf{K}_{u\varphi}^*\cdot\mathbf{U} - (1 + i\omega\zeta_d)^{-1}\mathbf{K}_{\varphi\varphi}\cdot\boldsymbol{\Phi} = \mathbf{F}_\varphi \tag{2.10}$$

3. THE METHOD OF EXPANSION IN MODES

Following the previously developed approach [3, 4, 6], we will now consider the special features of the use of the method of expansion in modes, which is the classical finite element method, in problems of electroelasticity. Here, the use of the model to take account of attenuation (1.5), (1.6) when $\beta_{dj} = \zeta_d$ will be a new aspect. Moreover, unlike in [3, 4, 6] where a problem of steady vibrations was considered, below we describe a method of expansion in modes for non-stationary problems.

Suppose $\{\omega_k, \mathbf{W}_k\}$ ($k = 1, 2, \ldots, n$, where n is the dimension of the matrices $\mathbf{M}_{uu}$ and $\mathbf{K}_{uu}$) are pairs of natural frequencies (electrical resonances) ω_k and the eigenvectors (normal modes) $\mathbf{W}_k$ corresponding to them, which are non-trivial solutions of homogeneous problem (2.9), (2.10) when there is no damping ($\alpha_{dj} = \beta_{dj} = \zeta_d = 0$). This problem is equivalent to the generalized eigenvalue problem

$$-\omega^2 \mathbf{M}_{uu} \cdot \mathbf{U} + \overline{\mathbf{K}}_{uu} \cdot \mathbf{U} = 0 \tag{3.1}$$

where

$$\overline{\mathbf{K}}_{uu} = \mathbf{K}_{uu} + \mathbf{K}_{u\varphi} \cdot \mathbf{K}_{\varphi\varphi}^{-1} \cdot \mathbf{K}_{u\varphi}^{*} \tag{3.2}$$

with symmetric matrices $\mathbf{M}_{uu} > 0$ and $\overline{\mathbf{K}}_{uu} \geqslant 0$. Since a matrix $\overline{\mathbf{K}}_{uu}$, which differs from the matrix $\mathbf{K}_{uu}$, occurs in equality (3.1), the eigenvectors $\mathbf{W}_k$ will now actually be orthogonal to the matrix $\overline{\mathbf{K}}_{uu}$, that is,

$$\mathbf{W}_m^* \cdot \overline{\mathbf{K}}_{uu} \cdot \mathbf{W}_k = \omega_m^2 \delta_{mk} \tag{3.3}$$

Note that the methods of determining the natural frequencies ω_k and the eigenvectors $\mathbf{W}_k$, which use algorithms for solving generalized problems in the case of sparse matrices as well as their implementation in the ACELAN finite-element package, have been described previously [3, 10].

We will assume that the conditions $\alpha_{dj} = \alpha_d$; $\beta_{dj} = \zeta_d$, for any j, are satisfied in the case of the attenuation factors α_{dj}, β_{dj}, and ζ_d of the initial model of a piezoelectric device (2.1), (1.5), (1.6). In the vectors $\mathbf{F}_u$ and $\mathbf{F}_\varphi$, we separate out the components $\mathbf{V}$ associated with the specified values of the electric potential. As a result, system (2.6), (2.7) can now be represented in the form

$$\mathbf{M}_{uu} \cdot (\ddot{\mathbf{U}} + \alpha_d \dot{\mathbf{U}}) + \mathbf{K}_{uu} \cdot (\mathbf{U} + \zeta_d \dot{\mathbf{U}}) + \mathbf{K}_{u\varphi} \cdot \mathbf{\Phi} = \mathbf{F}_{uv} \tag{3.4}$$

$$\mathbf{K}_{u\varphi}^* \cdot (\mathbf{U} + \zeta_d \dot{\mathbf{U}}) - \mathbf{K}_{\varphi\varphi} \cdot \mathbf{\Phi} = \mathbf{F}_{\varphi v} \tag{3.5}$$

where

$$\mathbf{F}_{uv} = \mathbf{F}_u - \mathbf{K}_{uv} \cdot \mathbf{V}, \quad \mathbf{F}_{\varphi v} = \mathbf{F}_\varphi + \zeta_d \dot{\mathbf{F}}_\varphi + \mathbf{K}_{\varphi v} \cdot \mathbf{V} \tag{3.6}$$

Note that the values of the electric potential, which appear in the expression for the vector $\mathbf{V}$, were assumed to be equal to zero in problem (3.1) and, hence, the corresponding degrees of freedom of the electric potential do not appear in the vector $\mathbf{\Phi}$.

From relation (3.5), $\mathbf{\Phi}$ can be expressed in terms of $\mathbf{U}$

$$\mathbf{\Phi} = \mathbf{K}_{\varphi\varphi}^{-1} \cdot \mathbf{K}_{u\varphi}^* \cdot (\mathbf{U} + \zeta_d \dot{\mathbf{U}}) + \mathbf{\Phi}_q \tag{3.7}$$

$\mathbf{\Phi}_q$ is actually determined from the separate "quasielectrostatic" problem

$$\mathbf{K}_{\varphi\varphi} \cdot \mathbf{\Phi} = -\mathbf{F}_{\varphi v} \tag{3.8}$$

Using expression (3.7), Eq. (3.4) can be rewritten in the form

$$\mathbf{M}_{uu} \cdot (\ddot{\mathbf{U}} + \alpha_d \dot{\mathbf{U}}) + \overline{\mathbf{K}}_{uu} \cdot (\mathbf{U} + \zeta_d \dot{\mathbf{U}}) = \mathbf{F}_{uv} - \mathbf{K}_{u\varphi} \cdot \mathbf{\Phi}_q \tag{3.9}$$

We shall seek the solution $\mathbf{U}$ of problem (3.9), (2.8) in the form of an expansion in modes

$$\mathbf{U} = \sum_{k=1}^{n} Z_k(t) \mathbf{W}_k \tag{3.10}$$

Substituting this expansion into Eq. (3.9), multiplying the resulting equality scalarly by $\mathbf{W}_m^*$ and using the relation for the orthogonality of the eigenvectors with respect to the matrices $\mathbf{M}_{uu}$ and $\overline{\mathbf{K}}_{uu}$, we obtain scalar differential equations for the individual functions $Z_k(t)$. Solving these equations, we find

$$Z_k = \frac{1}{\overline{\omega}_k}\int_0^t P_k(\tau)e^{-\xi_k\omega_k(t-\tau)}\sin[\overline{\omega}_k(t-\tau)]d\tau + A_k(0)e^{-\xi_k\omega_k t}\sin(\overline{\omega}_k t+\delta_k) \tag{3.11}$$

$$P_k = \mathbf{W}_k^*\cdot(\mathbf{F}_{uv} - \mathbf{K}_{u\varphi}\cdot\mathbf{\Phi}_q),\quad \xi_k = \frac{1}{2\omega_k}\alpha_d + \frac{\omega_k}{2}\zeta_d$$

$$\overline{\omega}_k = \omega_k(1-\xi_k^2)^{1/2},\quad A_k(0) = \left[Z_k^2(0) + \frac{(\dot{Z}_k(0)+\xi_k\omega_k Z_k(0))^2}{\overline{\omega}_k^2}\right]^{1/2}$$

$$\delta_k = \operatorname{arctg}\frac{Z_k(0)\overline{\omega}_k}{\dot{Z}_k(0)+\xi_k\omega_k Z_k(0)}$$

$$Z_k(0) = \mathbf{W}_k^*\cdot\mathbf{M}_{uu}\cdot\mathbf{U}_0,\quad \dot{Z}_k(0) = \mathbf{W}_k^*\cdot\mathbf{M}_{uu}\cdot\dot{\mathbf{U}}_0$$

Hence, the solution of problem (3.4)–(3.6), (2.8) using the method of expansion in modes is given by formulae (3.10) and (3.11) for $\mathbf{U}$, and (3.7) and (3.8) for $\mathbf{\Phi}$.

In the part involving finding the vector $\mathbf{U}$, the method which has been presented is identical in many respects to the classical version of the method of expansion in modes in linear dynamic problems of the mechanics of a deformable solid. The orthogonality of the vectors $\mathbf{W}_k$ with respect to the matrix $\overline{\mathbf{C}}_{uu} = \alpha_d\mathbf{M}_{uu} + \zeta_d\overline{\mathbf{K}}_{uu}$, that is ensured by the damping model which has been adopted, considerably facilitates the splitting of the equations here. The coefficients ξ_k in (3.11) are the damping factors of the modes, and, using them, it is possible to determine approximately the attenuation factors α_d and ζ_d for the whole piezoelectric device. The implementation of the method in problems of non-stationary oscillations is now obvious (also, see [6] and [3, 4] when $\zeta_d = \overline{\beta}_d$).

4. THE NEWMARK SCHEME FOR SOLVING NON-STATIONARY PROBLEMS

The method of expansion in modes requires equality of the damping parameters for the different media (that is, that relations (1.4) are satisfied) and, also, homogeneity of the main boundary conditions for the displacements $\mathbf{u}(\mathbf{x}, t)$. Methods of direct integration with respect to time are more general. In the same way as described earlier in [9], we shall use the Newmark method for integrating Cauchy problem (2.6)–(2.8) with respect to time in a formulation in which the velocities and accelerations in the time layers do not explicitly appear.

The usual Newmark scheme [1, 11] is based on the following expansions of the vector functions $\mathbf{a}_{i+1} = \mathbf{a}(t_{i+1})$, $\dot{\mathbf{a}}_{i+1} = \dot{\mathbf{a}}(t_{i+1})$, $\mathbf{a} = \{\mathbf{U}, \mathbf{\Phi}\}$ ($t_i = i\tau$; $\tau = \Delta t$ is the time step size)

$$\mathbf{a}_{i+1}^P = \mathbf{a}_i + \tau\dot{\mathbf{a}}_i + (1/2-\beta)\tau^2\ddot{\mathbf{a}}_i,\quad \mathbf{a}_{i+1} = \mathbf{a}_{i+1}^P + \beta\tau^2\ddot{\mathbf{a}}_{i+1} \tag{4.1}$$

$$\dot{\mathbf{a}}_{i+1}^P = \dot{\mathbf{a}}_i + (1-\gamma)\tau\ddot{\mathbf{a}}_i,\quad \dot{\mathbf{a}}_{i+1} = \dot{\mathbf{a}}_{i+1}^P + \gamma\tau\ddot{\mathbf{a}}_{i+1}$$

where β and γ are parameters of the Newmark method.

We now introduce the averaging operator Y_i

$$Y_j\,\mathbf{a} = \sum_{k=0}^{2}\beta_k\mathbf{a}_{j+1-k} \tag{4.2}$$

where

$$\beta_0 = \beta,\quad \beta_1 = \gamma_+ - 2\beta,\quad \beta_2 = \gamma_- + \beta;\quad \gamma_\pm = 1/2 \pm \gamma$$

The following lemma holds.

Lemma. Suppose the quantities $\mathbf{a}_i$, $\dot{\mathbf{a}}_i$, $\ddot{\mathbf{a}}_i$, $\mathbf{a}_i^p$ and $\dot{\mathbf{a}}_i^p$ are connected by relations (4.1) and (4.2) for all $i \in N$. The equalities

$$\begin{aligned} &\mathrm{Y}_i\,\mathbf{a}^p = \gamma_+\mathbf{a}_i + \gamma_-\mathbf{a}_{i-1};\quad \mathrm{Y}_i\,\dot{\mathbf{a}}^p = (\mathbf{a}_i - \mathbf{a}_{i-1})/\tau \\ &\mathrm{Y}_i\,\dot{\mathbf{a}} = (\gamma\mathbf{a}_{i+1} - (2\gamma-1)\mathbf{a}_i - (1-\gamma)\mathbf{a}_{i-1})/\tau \\ &\mathrm{Y}_i\,\ddot{\mathbf{a}} = (\mathbf{a}_{i+1} - 2\mathbf{a}_i + \mathbf{a}_{i-1})/\tau^2 \end{aligned} \tag{4.3}$$

then hold.

It is also easily shown that the quantities $\mathrm{Y}_i\ddot{\mathbf{a}}$ and $\mathrm{Y}_i\dot{\mathbf{a}}$ can be expressed in terms of $\mathrm{Y}_i\mathbf{a}^p$, $\mathrm{Y}_i\dot{\mathbf{a}}^p$ and $\mathrm{Y}_i\mathbf{a}$ using the formulae

$$\mathrm{Y}_i\,\ddot{\mathbf{a}} = \frac{1}{\beta\tau^2}(\mathrm{Y}_i\,\mathbf{a} - \mathrm{Y}_i\,\mathbf{a}^p),\quad \mathrm{Y}_i\,\dot{\mathbf{a}} = \frac{\gamma}{\beta\tau}(\mathrm{Y}_i\,\mathbf{a} - \mathrm{Y}_i\,\mathbf{a}^p) + \mathrm{Y}_i\,\dot{\mathbf{a}}^p \tag{4.4}$$

We now act on Eqs (2.6) and (2.7), which have been written at the instants of time t_i, with the averaging operator Y_i (4.2). When expressions (4.4) are taken into account, we obtain the following solving systems of linear algebraic equations for each time layer

$$\begin{aligned} &\mathbf{K}_{uu}^{\text{eff}}\cdot \mathrm{Y}_i\,\mathbf{U} + \mathbf{K}_{u\varphi}\cdot \mathrm{Y}_i\,\mathbf{\Phi} = \mathbf{F}_u^{\text{eff}} \\ &\mathbf{K}_{u\varphi}^{*}\cdot \mathrm{Y}_i\,\mathbf{U} - \left(1 + \frac{\zeta_d\gamma}{\beta\tau}\right)^{-1}\mathbf{K}_{\varphi\varphi}\cdot \mathrm{Y}_i\,\mathbf{\Phi} = F_\varphi^{\text{eff}} \end{aligned} \tag{4.5}$$

where

$$\mathbf{K}_{uu}^{\text{eff}} = \frac{1}{\beta\tau^2}\mathbf{M}_{uu} + \frac{\gamma}{\beta\tau}\mathbf{C}_{uu} + \mathbf{K}_{uu}$$

$$\mathbf{F}_u^{\text{eff}} = \mathrm{Y}_i\,\mathbf{F}_u + \mathbf{M}_{uu}\cdot \mathrm{Y}_i\,\mathbf{U}^p + \mathbf{C}_{uu}\cdot\left(\frac{\gamma}{\beta\tau}\mathrm{Y}_i\,\mathbf{U}^p - \mathrm{Y}_i\,\dot{\mathbf{U}}^p\right)$$

$$\mathbf{F}_\varphi^{\text{eff}} = \mathrm{Y}_i\,\mathbf{F}_\varphi + \zeta_d\left(1 + \frac{\zeta_d\gamma}{\beta\tau}\right)^{-1}\left[\frac{\gamma}{\beta\tau}\mathrm{Y}_i\,\mathbf{F}_\varphi^p - \mathrm{Y}_i\,\dot{\mathbf{F}}_\varphi^p + \mathbf{K}_{u\varphi}^{*}\cdot\left(\frac{\gamma}{\beta\tau}\mathrm{Y}_i\,\mathbf{U}^p - \mathrm{Y}_i\,\dot{\mathbf{U}}^p\right)\right]$$

expressions of the form of the first two relations of (4.3) hold for $\mathrm{Y}_i\mathbf{U}^p$, $\mathrm{Y}_i\dot{\mathbf{U}}^p$, $\mathrm{Y}_i\mathbf{F}_\varphi^p$ and $\mathrm{Y}_i\dot{\mathbf{F}}_\varphi^p$, expressions of the form (4.2) hold for $\mathrm{Y}_i\mathbf{F}_u$ and $\mathrm{Y}_i\mathbf{F}_\varphi$, and a further formula is required for the transition to the following time layer

$$\mathbf{a}_{i+1} = (\mathrm{Y}_i\,\mathbf{a} - \beta_1\mathbf{a}_i - \beta_2\mathbf{a}_{i-1})/\beta \tag{4.6}$$

which follows from (4.2).

The system of equations (4.5) in the vector of the averaged nodal unknowns $\{\mathrm{Y}_i\mathbf{U}, \mathrm{Y}_i\mathbf{\Phi}\}$ is written in symmetric form with the matrix $\mathbf{K}^{\text{eff}}$ of the saddle structure [9]

$$\mathbf{K}^{\text{eff}} = \begin{Vmatrix} \mathbf{K}_{uu}^{\text{eff}} & \mathbf{K}_{u\varphi} \\ \mathbf{K}_{u\varphi}^{*} & -(1 + \zeta_d\gamma/\beta\tau)^{-1}\mathbf{K}_{\varphi\varphi} \end{Vmatrix}$$

This matrix can be factorized using the square root method [9], and then only systems of linear algebraic equations with lower and upper triangular matrices can be solved in each time layer.

According to the lemma, the Newmark scheme presented here is mathematically equivalent to the usual Newmark scheme with velocities and accelerations [1, 11], and, consequently, it is absolutely stable when $\beta \geqslant (\frac{1}{2} + \gamma)^2/4$, $\gamma \geqslant \frac{1}{2}$ and, when $\beta \geqslant \frac{1}{4}$, $\gamma = \frac{1}{2}$, it does not have an approximation viscosity [11]. However, the Newmark scheme (4.5)–(4.6) does not explicitly use velocities and accelerations, which is preferable in the case of the electroelasticity problems considered here, when there are no velocities and accelerations of the electric potential in the equations.

Note that the Newmark method in its conventional formulation has been used to integrate the finite-element method equations with respect to time for plane electroelasticity problems, taking account of losses, in [12, 13] and in other papers by the same authors.

5. NUMERICAL EXPERIMENTS

The method for the finite-element analysis of harmonic and non-stationary problems which has been described has been implemented in the ACELAN package, that was specially designed to analyse piezoelectric devices [14]. In ACELAN, a set of partitioned saddle algorithms is used for the different finite-element procedures and for solving matrix problems in the finite element method [3, 9], and the schemes which have been described above are therefore embedded in an optimal manner in the overall ideology of the package.

As an example, consider a cylindrical rod of length $l = 0.040567$ m and radius $R = 0.002$ m. We relate the rod to a Cartesian system of coordinates $Ox_1x_2x_3$, directing the x_3 axis along the rod axis and locating the x_1 and x_2 axes in the plane of its lower end. Suppose the rod is made of PZT-4 piezoceramic, polarized along its length. The ends of the rod are assumed to be completely made into electrodes and a potential difference $\Delta\varphi = V(t)$ is applied to them. We shall assume that the lower end $x_3 = 0$ is rigidly damped and that the remaining faces of the rod are free from mechanical stresses.

In the case of the action of a harmonically varying potential difference $V = V_0 \exp[i2\pi ft]$ of frequency $f = \omega/(2\pi)$, we have a problem of steady vibrations. We shall consider frequencies f which are close to the first frequencies of electric resonance f_{r1} and antiresonance f_{a1}. It is well known that, for the case when the vibrations occur in piezorigid modes, the most important integral characteristic of the rod is the value of the electric impedance

$$Z = \frac{V}{I} = \frac{V}{i\omega Q}, \quad Q = -2\pi \int_0^R D_3 dr \quad \text{for} \quad x_3 = l$$

where I is the current and Q is the charge on the upper electrode.

Close to the first resonance frequencies it is possible to use the one-dimensional theory for a longitudinally polarized piezoelectric rod. In the approximation of this rod theory, in the case of the damping model (1.7), (1.8), (2.1), the following representation can be obtained for the electric impedance

$$Z = \frac{1 + i\zeta_d \omega}{i\omega C_0 (1 - k_{33}^2)} \left[1 - k_{33}^2 \frac{\operatorname{tg} \Omega l}{\Omega l} \right] \tag{5.1}$$

$$C_0 = \frac{\pi R^2 \varepsilon_{33}^T}{l}, \quad \Omega = \frac{1}{v_3^D} \sqrt{\frac{\omega^2 - i\alpha_d \omega}{1 + i\zeta_d \omega}}$$

The electromechanical coupling coefficient k_{33} and the longitudinal velocity v_3^D are determined using the usual formulae for electroelasticity [8].

We will also solve the problem of the harmonic vibrations of a rod which has been described numerically in an axisymmetric formulation using ACELAN.

We will determine the damping coefficients α_d and $\beta_d = \zeta_d$ of the model (1.7), (1.8), (2.1) by a method which uses the value of the Q-factor for two modes. We shall assume that the Q-factors of the rod at resonance frequencies of $f_{r1} = 18.867$ kHz and $f_{r2} = 73.345$ kHz are identical and equal to 500. The coefficients α_d and ζ_d can then be approximately calculated using the formulae in [3], which are the usual formulae in the case of the method of expansion in modes

$$\alpha_d = \frac{2\pi f_{r1} f_{r2}}{f_{r1} + f_{r2}}, \quad \zeta_d = \frac{1}{2\pi(f_{r1} + f_{r2})} \tag{5.2}$$

which gives $\alpha_d = 190\ \text{s}^{-1}$ and $\zeta_d = 0.345 \times 10^{-8}$ s.

The results of ACELAN calculations of the dependence of the real part (Re Z) and imaginary part (Im Z) of the electrical impedance Z on the frequency close to the first electrical antiresonance frequency f_{a1} are shown in Fig. 1 ($V_0 = 10^3$ V).

An unstructured finite-element mesh consisting of 90 triangular, square, piezoelectric, axisymmetric elements with a maximum diameter $h = 0.002$ m was used in the ACELAN calculations. Here, the total number of mesh points of the finite-element partitioning was equal to 229.

Graphs of Re Z and Im Z, obtained using formulae (5.1) for one-dimensional rod theory, are shown for comparison by the dashed curves in Fig. 1. The visual difference between the solid and dashed curves is explained by the small frequency step size. At the same time, the percentage difference between them is extremely small. For instance, the first antiresonance frequency, close to which the maximum in Re Z is observed, is equal to 25,039 Hz in the ACELAN calculations while, using the rod model, it is

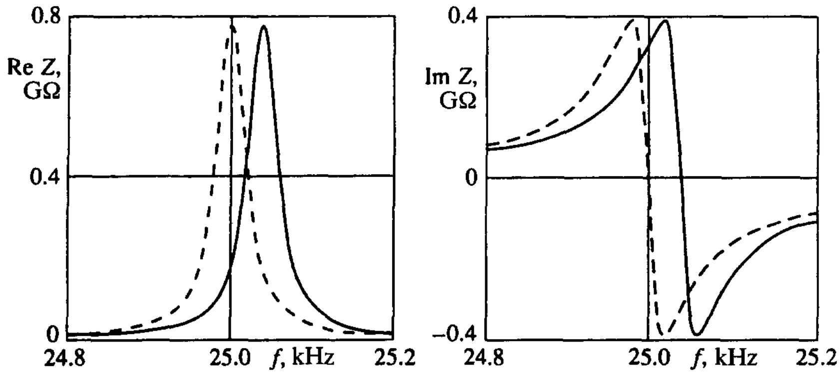

Fig. 1

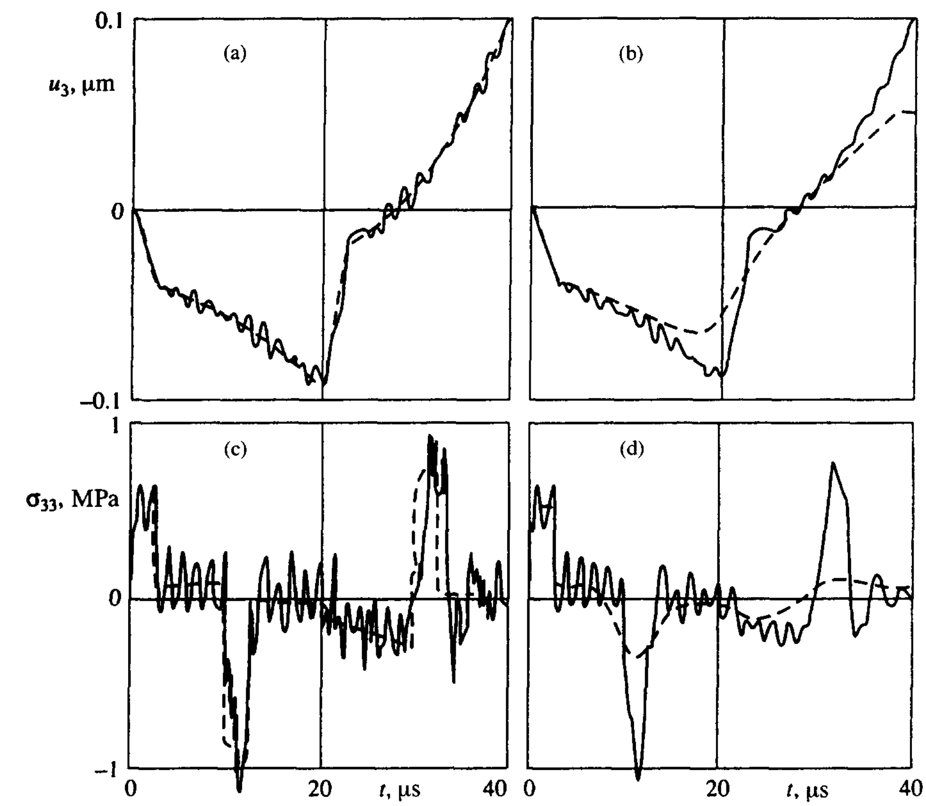

Fig. 2

25,000 Hz, that is, it is only 0.16% less. It can therefore be concluded that, in the neighbourhood of the first antiresonance frequency, rod theory enables one to determine the impedance Z of the piezoelectric transducer being considered with sufficient accuracy.

We also note that, in the three-dimensional or axisymmetric formulations for the model (1.7), (1.8), (2.1), it would be logical to specify the damping coefficients α_d and ζ_d as in the one-dimensional approximation. This enabled us to obtain practically identical amplitude values of Re Z and Im Z both in calculations of the axisymmetric problem using the finite element method and in calculations using the formulae of rod theory. Meanwhile, in the case of the model (1.7), (1.8), (2.1) when $\zeta_d = 0$, that is, the model adopted in ANSYS, the problem of determining the damping constants is considerably more difficult compared with one-dimensional problems. Some rather unwieldy formulae for calculating

α_d and β_d have been suggested in [3] (in this case, an error in formulae (5.2) for β_d in [3] has been admitted and, in order to remove this error, it is necessary to interchange the positions of the Q-factors Q_1 and Q_2).

We will now consider the same problem for a rod but in the non-stationary formulation. Suppose a potential difference $V(t)$ is applied to the electrodes of the rod in the form of the following impulse: $V = V_0[H(t) - H(t - t_0)]$, where $H(t)$ is the Heaviside function, $V_0 = 10^3$ V and $t_0 = 0.25 \times 10^{-5}$ s. The initial conditions (2.4) are taken to be null conditions.

When there is no attenuation, the non-stationary problem for the rod in the one-dimensional approximation has an exact solution which can be constructed, for example, by the method of reflected – refracted waves and, when account is taken of the first three reflections, this solution is given by formulae which contain the characteristic exponential terms

$$u_3(x_3, t) = \chi l u(z, t), \quad \sigma_{33}(x_3, t) = \chi (s_{33}^D)^{-1} \sigma(z, t)$$

$$\chi = V_0 \sqrt{s_{33}^D \varepsilon_{33}^S} / l, \quad z = x_3 / l, \quad \tau = t / T_0, \quad \tau_0 = t_0 / T_0, \quad T_0 = l / v_3^D$$

$$u(z, \tau) = k_{33}(F(z, \tau) - F(z, \tau - \tau_0)), \quad \tau < 4$$

$$\sigma(z, \tau) = u_{,z}(z, \tau) - \alpha u(1, \tau) - k_{33} H(\tau), \quad \alpha = k_{33}^2 \tag{5.3}$$

$$F(z, \tau) = \sum_{\pm} \pm H(\tau - 1 \pm z)(e^{\alpha(\tau - 1 \pm z)} - 1) +$$

$$+ \sum_{\pm} \pm H(\tau - 3 \pm z)\{[1 - 2\alpha(\tau - 3 \pm z)]e^{\alpha(\tau - 3 \pm z)} - 1\}$$

It can be seen from the structure of formulae (5.3) that the piezoelectric effect considerably complicates the wave pattern in the rod compared with the purely elastic case.

The non-stationary problem was also calculated in the axisymmetric formulation using ACELAN. The time step was taken to be equal to $\Delta t = 0.25 \times 10^{-6}$ s and the finite-element mesh was taken to be the same as in the problem of steady vibrations.

Graphs of the axial displacement u_3 at the centre of the upper end and the axial stresses σ_{33} at the centre of the lower end are shown in Fig. 2. The graphs in Fig. 2(a) and (c) were obtained ignoring damping, that is, when $\alpha_d = \beta_d = \zeta_d = 0$. The results of calculations for the axisymmetric problem are shown by the solid curves in Fig. 2(a) and (c), and the solution of the one-dimensional problem, constructed using formulae (5.3), is shown by the dashed curves. Since both the wave motions of the rod along its axis as well as the waves reflected from the end face are taken into account in the axisymmetric problem, the difference between the solid and the dashed curves is completely understandable. Nevertheless, it can be seen that, in the non-stationary case, the solutions obtained using the finite element method, as implemented by ACELAN and using rod theory, are found to be quite close.

ACELAN results, obtained taking damping into account in accordance with (1.7), (1.8) and (2.1), are shown in Fig. 2(b) and (d). The solid curves in Fig. 2(b) and (d) correspond to the results of calculations using the actual values of the damping coefficients $\alpha_d = 190$ s^{-1} and $\zeta_d = 0.345 \times 10^{-8}$ s for PZT-4 piezoceramics calculated using relations (5.2).

It is clear from a comparison of Fig. 2(b) and (d) and Fig. 2(a) and (c) that, in the problem considered, the actual attenuation for short times only slightly changes the wave pattern. On artificially increasing the coefficients α_d and ζ_d by a factor of 100, we obtain the significantly smoother graphs of the displacements and stresses with smaller amplitudes represented by the dashed curves in Figs 2(b) and (d).

The example presented above was also chosen with the aim of comparing the results of numerical calculations obtained using the ACELAN software package with the analytical solution. We see that, in the case of the same attenuation coefficients α_d and β_d ($\zeta_d = 0$), different ACELAN results were compared with the analogous results obtained using the ANSYS programme and practically identical results were obtained.

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